UNITARY–RATE HYBRID PETRI NETS

F. Balduzzi, A. Di Febbraro*
A. Giua, C. Seatzu**

* Dip. di Automatica ed Informatica, Politecnico di Torino, Italy
email: {balduzzi,difebbraro}@polito.it.
** Dip. di Ingegneria Elettrica ed Elettronica, Università di Cagliari, Italy
email: {giua,seatzu}@diee.unica.it.

Abstract. In this paper we deal with a hybrid formalism based on Petri nets. A restricted model, called Unitary Rate Hybrid Petri Net, is defined. This model can be seen as the Petri net counterpart of a Timed Automaton. We demonstrate that the reachability problem for a hybrid net in this class can be reduced to the reachability problem of a corresponding discrete Petri net, and thus it is decidable.

1. Introduction

The control of hybrid systems, i.e., systems with both time–driven and event–driven dynamics, is a domain of increasing importance and several hybrid models have been presented in the literature.

Petri nets (PNs) [4] have originally been introduced to describe and analyze discrete event systems. Recently, much effort has been devoted to apply these models to hybrid systems as well. Among the many different hybrid net formalisms that have been proposed, we consider here a basic model that was originally presented in [2] and that was inspired from the approach of David and Alla [3]. This model, that will be called in the rest of this paper *Hybrid Petri Net* (HPN), consists of continuous places holding fluid, discrete places containing a non–negative integer number of tokens, and transitions, either discrete or continuous. Note that, unlike [2], we are assuming here that no timing structure is associated to the firing of discrete transitions.

In this paper we define a particular class of HPS called *unitary–rate HPN* (URHPN), that can be seen as the HPN counterpart of a Timed Automaton (TA) [1]. It consists of a HPN where the continuous dynamics is such that the marking of each continuous place constantly increases with a *unitary* slope. Thus the marking of each continuous place represents the value of a timer. When comparing URHPNs and TA we observe that: TA can model "reset" of the continuous state, while URHPNs can model "jumps of constant magnitude" of the continuous state (and, as in the general case, may also have an infinite discrete state space) [6].

We prove that the reachability problem is decidable for a URHPN and can be reduced to the reachability problem of a discrete PN with a suitable initial marking. This result may not be surprising, because the reachability problem is also known to be decidable for TA [1].

2. Hybrid Petri Nets

The Petri net formalism used in this paper can be seen as the "untimed" version of the model presented in [2]. For a more comprehensive introduction to place/transition Petri nets see [4].

A Hybrid Petri Net (HPN) is a structure $N = (P, T, Pre, Post, \mathcal{C})$.

The set of *places* $P = P_d \cup P_c$ is partitioned into a set of *discrete* places P_d (represented as circles) and a set of *continuous* places P_c (represented as double circles). The cardinality of P, P_d and P_c is denoted n, n_d and n_c.

The set of *transitions* $T = T_d \cup T_c$ is partitioned into a set of discrete transitions T_d and a set of continuous transitions T_c (represented as double boxes). The cardinality of T, T_d and T_c is denoted q, q_d and q_c.

The pre- and post-incidence functions that specify the arcs are (here $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$): $Pre : P_d \times T \to \mathbb{N}$, $Post : P_c \times T \to \mathbb{R}_0^+$.

We require (well–formed nets) that for all $t \in T_c$ and for all $p \in P_d$, $Pre(p,t) = Post(p,t)$.

The function $\mathcal{C} : T_c \to \mathbb{R}_0^+ \times \mathbb{R}_\infty^+$ specifies the firing speeds associated to continuous transitions (here $\mathbb{R}_\infty^+ = \mathbb{R}^+ \cup \{\infty\}$). For any continuous transition $t_j \in T_c$ we let $\mathcal{C}(t_j) = (V_j', V_j)$, with $V_j' \leq V_j$. Here V_j' represents the minimum firing speed (mfs) and V_j represents the maximum firing speed (MFS).

We denote the preset (postset) of transition t as ${}^\bullet t$ ($t^\bullet$) and its restriction to continuous or discrete places as ${}^{(d)}t = {}^\bullet t \cap P_d$ or ${}^{(c)}t = {}^\bullet t \cap P_c$. Similar notation may be used for presets and postsets of places. The incidence matrix of the net is defined as $\boldsymbol{C}(p,t) = Post(p,t) - Pre(p,t)$. The restriction of $\boldsymbol{C}$ to P_X and T_Y $(X, Y \in \{c, d\})$ is denoted $\boldsymbol{C}_{XY}$. Note that by the well-formedness hypothesis $\boldsymbol{C}_{dc} = 0$.

A marking $\boldsymbol{m} : P_d \to \mathbb{N},\ P_c \to \mathbb{R}_0^+$ is a function that assigns to each discrete place a non-negative number of tokens, represented by black dots and assigns to each continuous place a fluid volume; m_p denotes the marking of place p. The value of a marking at time τ is denoted $\boldsymbol{m}(\tau)$. The restriction of $\boldsymbol{m}$ to P_d and P_c are denoted with $\boldsymbol{m}^d$ and $\boldsymbol{m}^c$, respectively. An HPN system $(N, \boldsymbol{m}(\tau_0))$ is an HPN N with an initial

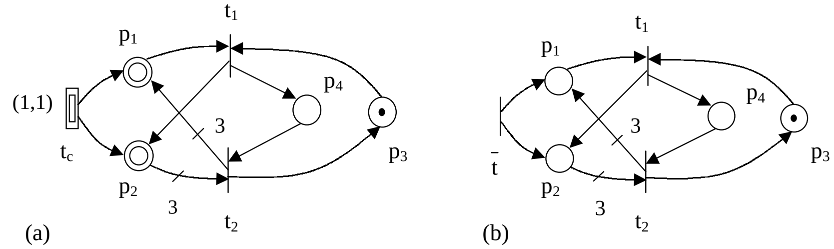


Figure 1: *(a) A URHPN; (b) the corresponding discretized PN.*

marking $\boldsymbol{m}(\tau_0)$.

The enabling of a discrete transition depends on the marking of all its input places, both discrete and continuous.

Definition 1. *Let $(N, \boldsymbol{m})$ be an HPN system. A discrete transition t is enabled at $\boldsymbol{m}$ if for all $p_i \in {}^\bullet t$, $m_i \geq Pre(p_i, t)$.* ■

A continuous transition is enabled only by the marking of its input discrete places. The marking of its input continuous places, however, is used to distinguish between strongly and weakly enabling.

Definition 2. *Let $(N, \boldsymbol{m})$ be an HPN system. A continuous transition t is enabled at $\boldsymbol{m}$ if for all $p_i \in {}^{(d)}t$, $m_i \geq Pre(p_i, t)$.*

We say that an enabled transition $t \in T_c$ is: strongly enabled at $\boldsymbol{m}$ if for all places $p_i \in {}^{(c)}t$, $m_i > 0$; weakly enabled at $\boldsymbol{m}$ if for some $p_i \in {}^{(c)}t$, $m_i = 0$.

In the following we describe the hybrid dynamics of an HPN. We first consider the time–driven behavior associated to the firing of continuous transitions, and then the event–driven behavior associated to the firing of discrete transitions.

The instantaneous firing speed (IFS) at time τ of a transition $t_j \in T_c$ is denoted $v_j(\tau)$. We can write the equation which governs the evolution in time of the marking of a place $p_i \in P_c$ as

$$\dot{m}_i(\tau) = \sum_{t_j \in T_c} C(p_i, t_j) v_j(\tau) \tag{1}$$

where $\boldsymbol{v}(\tau) = [v_1(\tau), \ldots, v_{n_c}(\tau)]^T$ is the IFS vector at time τ. Indeed Equation 1 holds assuming that at time τ no discrete transition is fired and that all speeds $v_j(\tau)$ are continuous in τ.

The enabling state of a continuous transition t_j defines its admissible IFS v_j. If t_j is not enabled then $v_j = 0$. If t_j is strongly enabled, then it may fire with any firing speed $v_j \in [V'_j, V_j]$. If t_j is weakly enabled, then it may fire with any firing speed $v_j \in [V'_j, \overline{V}_j]$, where $\overline{V}_j \leq V_j$ since t_j cannot remove more fluid from any empty input continuous place $\overline{p}$ than the quantity entered in $\overline{p}$ by other transitions.

We now characterize the set of all admissible IFS vectors.

Definition 3. (admissible IFS vectors)
Let $(N, \boldsymbol{m})$ be an HPN system. Let $T_{\mathcal{E}}(\boldsymbol{m}) \subset T_c$ ($T_{\mathcal{N}}(\boldsymbol{m}) \subset T_c$) be the subset of continuous transitions enabled (not enabled) at $\boldsymbol{m}$, and $P_{\mathcal{E}} = \{p_i \in P_c \mid m_i = 0\}$ be the subset of empty continuous places. Any admissible IFS vector $\boldsymbol{v}$ at $\boldsymbol{m}$ is a feasible solution of the following linear set:

$$\left\{ \begin{array}{lll} (a) & V_j - v_j \geq 0 & \forall t_j \in T_{\mathcal{E}}(\boldsymbol{m}) \\ (b) & v_j - V'_j \geq 0 & \forall t_j \in T_{\mathcal{E}}(\boldsymbol{m}) \\ (c) & v_j = 0 & \forall t_j \in T_{\mathcal{N}}(\boldsymbol{m}) \\ (d) & \sum_{t_j \in T_{\mathcal{E}}} C(p, t_j) v_j \geq 0 & \forall p \in P_{\mathcal{E}}(\boldsymbol{m}). \end{array} \right. \tag{2}$$

The set of all feasible solutions is denoted $\mathcal{S}(N, \boldsymbol{m})$. ■

Constraints of the form (2.a), (2.b), and (2.c) follow from the firing rules of continuous transitions. Constraints of the form (2.d) follow from (1), because if a continuous place is empty then its fluid content cannot decrease.

Note that the set $\mathcal{S}$ is a function of the marking of the net. Thus as $\boldsymbol{m}$ changes it may vary as well. In particular it changes at the occurrence of the following macro–events: (a) a discrete transition fires, thus changing the discrete marking and enabling/disabling a continuous transition; (b) a continuous place becomes empty, thus changing the enabling state of a continuous transition from strong to weak.

Let τ_k and τ_{k+1} be the occurrence times of two consecutive macro–events of this kind; we assume that within the interval of time $[\tau_k, \tau_{k+1})$ the IFS vector is constant and we denote it $\boldsymbol{v}(\tau_k)$. Then the continuous behavior of an HPN for $\tau \in [\tau_k, \tau_{k+1})$ is described by: $\boldsymbol{m}^c(\tau) = \boldsymbol{m}^c(\tau_k) + \boldsymbol{C}_{cc}\boldsymbol{v}(\tau_k)(\tau - \tau_k)$, $\boldsymbol{m}^d(\tau) = \boldsymbol{m}^d(\tau_k)$.

The firing of a discrete transition t_j at $\boldsymbol{m}(\tau)$ yields the marking: $\boldsymbol{m}^c(\tau) = \boldsymbol{m}^c(\tau^-) + \boldsymbol{C}_{cd}\boldsymbol{\sigma}(\tau)$, $\boldsymbol{m}^d(\tau) = \boldsymbol{m}^d(\tau^-) + \boldsymbol{C}_{dd}\boldsymbol{\sigma}(\tau)$, where $\boldsymbol{\sigma}(\tau)$ is the firing count vector associated to the firing of transition t_j.

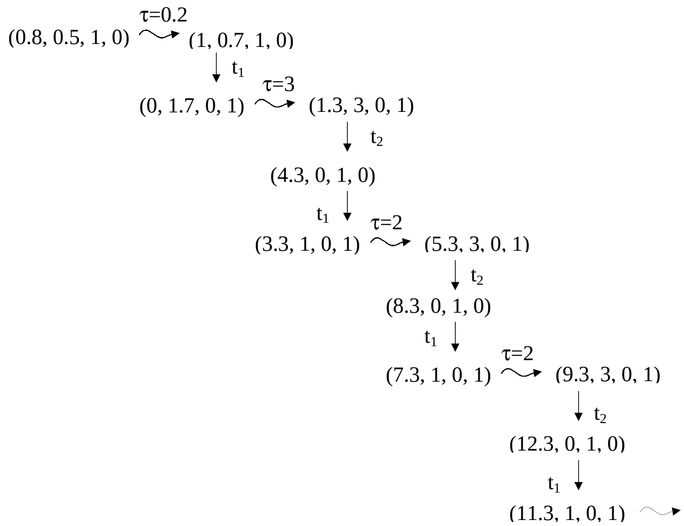


Figure 2: *The reachability graph of the URHPN in figure 1*

2.1. Firing sequence and reachability

Now, we provide some definitions that will be useful in the following.

Definition 4. *(Event Step) Let us consider a HPN system* $(N, \boldsymbol{m})$*. If* $t \in T_d$ *is enabled at* $\boldsymbol{m}$*,* t *may fire. The firing of* t *determines a new marking* $\tilde{\boldsymbol{m}} = \boldsymbol{m} + Post(\cdot, t) - Pre(\cdot, t)$ *and we write* $\boldsymbol{m}[t\rangle\tilde{\boldsymbol{m}}$*.* ■

We can use a similar notation for the marking variation due to the firing of continuous transitions.

Definition 5. *(Time Step) Let us consider a HPN system* $(N, \boldsymbol{m})$*. If* $t \in T_c$ *is enabled at* $\boldsymbol{m}$ *for a time interval of length* $\overline{\tau} \in \mathbb{R}^+$*. The firing of* t *during that time interval determines a new marking* $\tilde{\boldsymbol{m}}$*:* $\tilde{\boldsymbol{m}}^d = \boldsymbol{m}^d$*,* $\tilde{\boldsymbol{m}}^c = \int_0^{\overline{\tau}} \boldsymbol{C}_{cc}\boldsymbol{v}(\tau)d\tau + \boldsymbol{m}^c \geq \boldsymbol{0}$*, where* $\boldsymbol{v} \in \mathcal{S}(N, \boldsymbol{m})$ *and we write* $\boldsymbol{m}[\overline{\tau}\rangle\tilde{\boldsymbol{m}}$*.* ■

Definition 6. *Let* $(N, \boldsymbol{m})$ *be a HPN system. A firing sequence* $\sigma = \alpha_1, \cdots, \alpha_k \in (T_d \cup \mathbb{R}^+)^*$ *is enabled from a marking* $\boldsymbol{m}$ *if* $\boldsymbol{m}[\alpha_1\rangle\boldsymbol{m}_1[\alpha_2\rangle\boldsymbol{m}_2 \cdots [\alpha_k\rangle\tilde{\boldsymbol{m}}$ *holds. To denote that the firing of* σ *from* $\boldsymbol{m}$ *determines the marking* $\tilde{\boldsymbol{m}}$ *we write* $\boldsymbol{m}[\sigma\rangle\tilde{\boldsymbol{m}}$*.* ■

3. Unitary–rate hybrid Petri nets

In this section we define a special class of hybrid Petri nets called *unitary–rate* HPNs that can be seen as the net counterpart of timed automata.

Definition 7. *A unitary–rate hybrid Petri net (URHPN) is a HPN where:* $T_c = \{t_c\}$*,* ${}^{\bullet}t_c = \emptyset$*,* $\mathcal{C}(t_c) = (1, 1)$*,* $\forall p \in P_c : Post(p, t_c) = 1$*,* $Pre, Post \in \mathbb{N}^{n \times q}$*.* ■

Thus a unitary–rate hybrid Petri net has a *single* continuous transition that is always enabled — because it has no input places — and whose firing speed is always unitary. The marking of all continuous places increases with unitary rate during a time step. Discontinuous variations of continuous markings may only follow the firing of discrete transitions. Furthermore, we assume that all arcs have integer weights. Such an assumption has been introduced for simplicity. In fact, whenever $Pre, Post \in \mathbb{R}^{n \times q}$ all the weights could be multiplied by the least common multiple of the denominators of all the constants appearing in $Pre, Post$ to get a new hybrid net that is isomorphic with a new one where $Pre, Post \in \mathbb{N}^{n \times q}$. Even if $Pre, Post \in \mathbb{R}^{n \times q}$ but each weight has the same irrational numbers as common factors, an isomorphism with a net where $Pre, Post \in \mathbb{N}^{n \times q}$ can be determined.

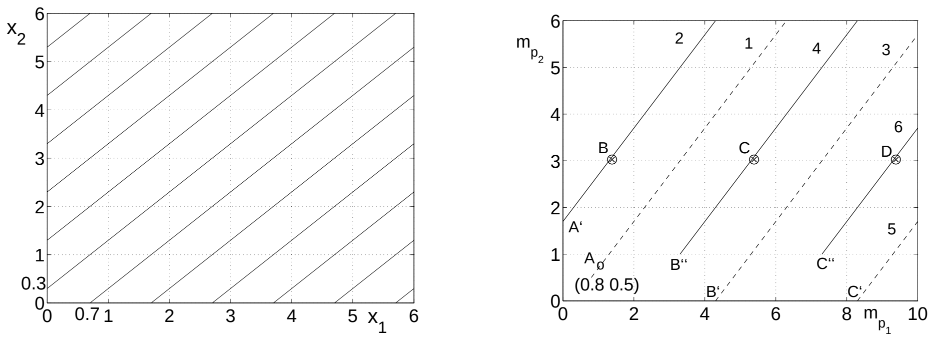
The evolution of URHPNs can be related to that of timed HA. In fact, the constant rate variation of continuous marking in URHPNs agrees with the set *Inclusions* containing the single element $\mathbf{1} \in \mathbb{R}^n$ in timed HA. However, all the differences outlined in the previous section still hold. In particular, in URHPNs the firing of a discrete transition may only produce constant variations on the continuous marking. On the other hand, URHPNs can assume an infinite number of discrete states.

Example 8. The HPN in figure 1.a is a URHPN. Its reachability graph is shown in figure 2 under the assumption that $\boldsymbol{m}_0 = (0.8, 0.5, 1, 0)$. It has been drawn in accordance with the following rule. The firing of the continuous transition is represented only if it produces a variation on the enabling condition of the net. Note however that the continuous transition is always enabled and always fires with a constant unitary rope. Therefore, all the markings obtained from those in figure 2 with the addition of the same positive real number to m_{p_1} and m_{p_2}, are reachable. ■

Now, we prove that the reachability problem for URHPNs is decidable.

Let us first define an equivalence relation on $(\mathbb{R}_0^+)^m$.

Definition 9. *A vector* $\boldsymbol{x} \in (\mathbb{R}_0^+)^m$ *is time-consistent with* $\boldsymbol{y} \in (\mathbb{R}_0^+)^m$ *if:* $\exists\, b \in [0, 1) : \forall i = 1, \cdots, m$, $\langle y_i \rangle = \langle x_i + b \rangle$ *where* $\langle \cdot \rangle$ *denotes the fractional part and we write* $\boldsymbol{x} \sim \boldsymbol{y}$*. The equivalence classes of this relation are denoted* $[\boldsymbol{x}]$*.* ■



(a) (b)

Figure 3: *(a) The equivalence class* $[(0,0.3)]$*; (b) the set of continuous markings for the URHPN in example 12.*

Example 10. Let $\boldsymbol{x} = (0, 0.3)$. In figure 3.a the set of vectors time-consistent with $\boldsymbol{x}$ are represented in the plane (x_1, x_2) and lie on a family of parallel lines. All lines are equally spaced and are characterized by a constant unitary slope. ■

Lemma 11. *Let* $(N, \boldsymbol{m})$ *be a URHPN system. If* $\tilde{\boldsymbol{m}} \in R(N, \boldsymbol{m})$ *then* $\tilde{\boldsymbol{m}}^c \in [\boldsymbol{m}^c]$*.*

Proof. If $\tilde{\boldsymbol{m}} \in R(N, \boldsymbol{m})$, then there exists a firing sequence $\sigma = \alpha_1, \alpha_2, \cdots, \alpha_k$ such that $\boldsymbol{m}[\alpha_1\rangle \boldsymbol{m}_1[\alpha_2\rangle \boldsymbol{m}_2 \cdots [\alpha_k\rangle \tilde{\boldsymbol{m}}$. Since all the arc weights are integers, the firing of a discrete transition produces no variation on the fractional parts of a continuous marking. Thus, if $\boldsymbol{m}_{i-1}[\alpha_i\rangle \boldsymbol{m}_i$ and $\alpha_i \in T_d$, then $\langle \boldsymbol{m}_{i-1} \rangle = \langle \boldsymbol{m}_i \rangle$ and $\boldsymbol{m}_i^c \in [\boldsymbol{m}_{i-1}^c]$.

On the contrary, the firing of the continuous transition produces a variation on the fractional parts of the continuous marking. However, all these variations have the same magnitude. Thus, if $\alpha_i = \overline{\tau} \in \mathbb{R}^+$, then $\boldsymbol{m}_i^c = \int_0^{\overline{\tau}} \boldsymbol{C}_{cc} v(\tau) d\tau + \boldsymbol{m}_{i-1}^c$. However, $v(\tau) = 1$ and $\boldsymbol{C}_{cc} = \mathbf{1}$ by hypothesis, hence $\boldsymbol{m}_i^c = \boldsymbol{m}_{i-1}^c + \overline{\boldsymbol{\tau}}$ where $\overline{\boldsymbol{\tau}}$ is a vector $\in \mathbb{R}^{n_c}$ whose components are all equal to $\overline{\tau}$. Now, let $b = \langle \overline{\tau} \rangle$, then $\forall p \in P_c, \langle m_{i,p} \rangle = \langle m_{i-1,p} + b \rangle$. Thus, $\boldsymbol{m}_i^c \in [\boldsymbol{m}_{i-1}^c]$.

Finally, we can conclude that $\tilde{\boldsymbol{m}}^c \in [\boldsymbol{m}^c]$ by the transitivity of equivalence relations. □

Example 12. Let us consider the URHPN system $(N, \boldsymbol{m}_0)$ in example 8 with initial marking $\boldsymbol{m}_0 = (0.8, 0.5, 1, 0)$. In figure 3.b the set of all continuous markings reachable from $\boldsymbol{m}_0$ is represented. Obviously, this is a subset of $[\boldsymbol{m}_0^c]$.

Lines have been partitioned in two different sets and distinguished as dash and continuous lines. Dash lines belong to the set of continuous markings reachable when the discrete marking is equal to $\boldsymbol{m}^d = (1, 0)$, while continuous lines belong to the set of continuous markings reachable in the case of $\boldsymbol{m}^d = (0, 1)$. The discrete marking changes every times one of the discrete transition fires and discrete transitions can only fire alternatively.

Let us examine all possible evolutions of the net when the initial marking is $\boldsymbol{m}_0$. During the first 0.2 time instants, no discrete transition is enabled and t_c fires until the marking moving along line 1 reaches point A corresponding to $(1, 0.7, 1, 0)$. Now t_1 become enabled. Thus from point A it may fire changing the marking to point A‘. Note however that t_1 is not required to fire as soon as A is reached; it may fire from any other point on line 1 greater than A thus reaching a corresponding point on line 2. For all markings on line 2 smaller than B no discrete transition is enabled and only the continuous transition fires until B is reached. Now t_2 become enabled. Thus from point B it may fire changing the marking to point B‘. Note however that t_2 is not required to fire as soon as B is reached; it may fire from any other point on line 2 greater than B thus reaching a corresponding point on line 3. All markings on line 3 enable transition t_1 that may fire thus reaching a corresponding point on line 4. Everything repeats periodically as shown in figure 3.b. We also observe that the points $A, A‘, B$, etc. that characterize the net evolution correspond to the markings in the reachability graph of figure 2.

■

Now, let us define a transformation on a hybrid Petri net system.

Definition 13. *Given a HPN* $N = (P, T, Pre, Post, \mathcal{C})$*. We define the "discretized PN associated to N", the P/T net* $\lfloor N \rfloor = (P', T', Pre', Post')$ *with:* $P' = P$*, i.e.,* $\lfloor N \rfloor$ *has as many places as N, but they are all discrete;* $T' = T$*, i.e.,* $\lfloor N \rfloor$ *has as many transitions as N, but they are all discrete;* $Pre'(p,t) = \lfloor Pre(p,t) \rfloor$*;* $Post'(p,t) = \lfloor Post(p,t) \rfloor$*, where* $\lfloor \cdot \rfloor$ *denotes the integer part. We call* $\lfloor N \rfloor$ *the discretized HPN associated to N.* ■

Example 14. In figure 1.b the discretized PN corresponding to the HPN in figure 1.a is shown. ■

Now, we provide a necessary and sufficient condition for a marking $\boldsymbol{m}$ in a URHPN to be reachable.

Theorem 15. *Let* $(N, \boldsymbol{m}_0)$ *be a URHPN system. Then,* $\boldsymbol{m} \in R(N, \boldsymbol{m}_0)$ *iff* $\boldsymbol{m}^c \in [\boldsymbol{m}_0^c]$ *and* $\lfloor \boldsymbol{m} \rfloor \in$

$R(\lfloor N \rfloor, \tilde{\boldsymbol{m}})$ *where*

$$\tilde{m}_p = \begin{cases} \lfloor m_{0,p} \rfloor + 1 & \text{if } \langle m_p \rangle < \langle m_{0,p} \rangle \\ \lfloor m_{0,p} \rfloor & \text{otherwise} \end{cases}$$

and $\lfloor N \rfloor$ *is the discretized net associated to* N.

Proof. First, let us observe that $\boldsymbol{m} \in R(N, \boldsymbol{m}_0)$ iff $\exists\, \sigma$ such that $\boldsymbol{m}_0[\sigma\rangle\boldsymbol{m}$. Since the continuous transition in $(N, \boldsymbol{m}_0)$ is always enabled, this implies that $\exists\, \sigma' = \sigma_\tau \sigma_T$ such that $\boldsymbol{m}_0[\sigma'\rangle\boldsymbol{m}$, where $\sigma_\tau \in \mathbb{R}_0^+$ and $\sigma_T \in T_d^*$, i.e., if $\boldsymbol{m}$ is reachable, then it may also be reached by a "normalized sequence" where a single time step occurs first, and all the event steps occur only at the end.

The firing sequence σ_τ can be written as $\sigma_\tau = \sigma'_\tau \sigma''_\tau$, where $\sigma'_\tau = \langle \sigma_\tau \rangle$, and $\sigma''_\tau = \lfloor \sigma_\tau \rfloor$. Therefore, $\boldsymbol{m}_0[\sigma'_\tau\rangle\boldsymbol{m}'_0[\sigma''_\tau\rangle\boldsymbol{m}'[\sigma_T\rangle\boldsymbol{m}$. Obviously, $\langle \boldsymbol{m}'_0 \rangle = \langle \boldsymbol{m}' \rangle = \langle \boldsymbol{m} \rangle$.

We now observe that the difference in the fractional part between $\boldsymbol{m}_0$ and $\boldsymbol{m}$ is due to the time step σ'_τ, that has a length less than one and yields $\boldsymbol{m}'_0$ from $\boldsymbol{m}_0$. Obviously, $\forall p \in P_c$, if $\langle m_p \rangle \equiv \langle m'_{0,p} \rangle \geq \langle m_{0,p} \rangle$ then $\lfloor m'_{0,p} \rfloor = \lfloor m_{0,p} \rfloor$. Otherwise, if $\langle m_p \rangle \equiv \langle m'_{0,p} \rangle < \langle m_{0,p} \rangle$, then $\lfloor m'_{0,p} \rfloor = \lfloor m_{0,p} \rfloor + 1$. Thus the integer part of $\boldsymbol{m}'_0$ is exactly the marking $\tilde{\boldsymbol{m}}$ defined in the theorem statement.

Finally we observe that because $\boldsymbol{m}'_0$ and $\boldsymbol{m}$ have the same fractional part, then $\boldsymbol{m} \in R(N, \boldsymbol{m}'_0)$ if and only if $\lfloor \boldsymbol{m} \rfloor \in R(\lfloor N \rfloor, \lfloor \boldsymbol{m}'_0 \rfloor)$. In fact, let $\bar{t}$ be the discrete transition of $\lfloor N \rfloor$ corresponding to the continuous transition t_c of N. With the notation used above it is easy to understand that $\boldsymbol{m}'_0[\sigma''_\tau \sigma_T\rangle\boldsymbol{m}$ if and only if $\lfloor \boldsymbol{m}'_0 \rfloor[\sigma''_T \sigma_T\rangle\lfloor \boldsymbol{m} \rfloor$ where σ''_T contains the transition $\bar{t}$ an number of times equal to σ''_τ. Thus, $\lfloor N \rfloor$ simulates N firing $\bar{t}$ for each time step of length 1 occurring in N. □

Example 16. Let us consider the URHPN system (N, m_0) in example 8 with initial marking $\boldsymbol{m}_0 = (0.8, 0.5, 1, 0)$. We want to determine if $\boldsymbol{m} = (5, 0.7, 1, 0) \in R(N, \boldsymbol{m}_0)$ by applying theorem 15.

Clearly $\boldsymbol{m}^c \in [\boldsymbol{m}_0^c]$ because if we take $b = 0.2$, then $\forall p \in P_c$, $\langle m_p \rangle = \langle m_{0,p} + b \rangle$. Then, if we consider the discretized PN in figure 1.b we see that $(5, 0, 1, 0) \in R(\lfloor N \rfloor, (1, 0, 1, 0))$ where, in accordance with the notation of theorem 15, $(1, 0, 1, 0) = \tilde{\boldsymbol{m}}$ and $(5, 0, 1, 0) = \lfloor \boldsymbol{m} \rfloor$. In fact, the firing sequence $\overline{\sigma} = \bar{t}, t_1, \bar{t}, t_2$ is such that $\tilde{\boldsymbol{m}}[\overline{\sigma}\rangle\lfloor \boldsymbol{m} \rfloor$. Therefore, we can conclude that even $\boldsymbol{m} \in R(N, \boldsymbol{m}_0)$. The same conclusion can be reached by looking at figure 3. In fact, it is easy to observe that the firing sequence $\sigma = 0.2, t_1, 1.3, t_2, 0.7$ is such that $\boldsymbol{m}_0[\sigma\rangle\boldsymbol{m}$. ■

By virtue of the above theorem 15, the results on the reachability of discrete Petri nets can be extended to URHPNs, thus proving the validity of the following corollary.

Corollary 17. *The reachability problem is decidable for URHPNs.*

Proof. Follows from theorem 15 and from the fact that the reachability problem is decidable for discrete PN [5]. □

4. Conclusions

In this paper we have defined a special class of Hybrid Petri Nets, called *Unitary Rate Hybrid Petri Nets*, that can be seen as the Petri net counterpart of a Timed Automaton. The reachability problem for a hybrid net in this class has been reduced to the reachability problem of a corresponding discrete Petri net, and thus it is decidable.

To study this class of nets, in one of the examples we have informally used the reachability graph analysis that has been developed for discrete nets. It may be interesting to find out if a technique based on the reachability/coverability graph may always be applied to this hybrid model and which properties can be studied with it.

It is also worth defining and exploring new restricted classes of HPNs. These structures may extend the classes of models for which important properties can be shown to be decidable and can be studied with standard tools of discrete Petri nets.

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