Chaotic Behaviour in Hybrid Systems

Barbara Cannas, Alessandra Fanni, Alessandro Giua, Carla Seatzu

Department of Electrical and Electronic Engineering

University of Cagliari, Piazza d'Armi — 09123 Cagliari, Italy

{*cannas,fanni,giua,seatzu*}*@diee.unica.it*

Abstract

In this paper we investigate whether and how deterministic hybrid automata may exhibit a chaotic behaviour. We present some significant examples showing that chaos may both originate from the non–linearity of the continuous evolution and from the discontinuity in the discrete state. In particular, we are interested in some special structures of the non–linearity, i.e., we consider hybrid automata whose activities within each location are linear (or affine functions) but as the location changes these activities change as well. Two examples of electrical circuits are also discussed.

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1 Introduction

There is no widely accepted definition of chaos.

From a practical point of view, it can be defined as a bounded steady state behaviour that is not an equilibrium point, not periodic, and not quasi–periodic [15]. Roughly speaking, from a mathematical point of view, chaos can be seen as an effectively unpredictable long time behaviour arising in a deterministic dynamical system because of sensitivity to initial conditions. It must be emphasized that a deterministic dynamical system is perfectly predictable given perfect knowledge of the initial conditions, and is in practice always predictable in the short term.

The study of chaotic behaviour has been usually focused on time–driven continuous systems, where chaos derives from the special structure of the non–linearity in the dynamics. Time–driven discontinuous systems have also been studied — albeit less deeply — and the chaotic behaviour of these systems has been explored as well.

Hybrid systems, i.e., systems that combine time–driven and event–driven dynamics, are a generalization of time–driven systems, thus it is obvious that they may have chaotic behavior.

The most well known hybrid model is the Hybrid Automaton (HA) [9, 13, 16], a generalization of the Timed Automaton defined by Alur and Dill [2]. A HA consists of a classical automaton provided with a continuous state and a finite number of discrete states called locations. In each location the state is constrained to belong to a given set (invariant), and may evolve in time with different dynamics within a differential inclusion (activity). The continuous state may also have discontinuous jumps at the occurrence of a discrete event belonging to a predefined set of feasible events (guard).

The recent results on HA have shown that a trade–off between modeling power and analytical tractability is necessary. To this end, several special classes of HA have been studied (by posing some structural restrictions on guards and activities): timed automata, timed automata with skewed clocks, multirate and rectangular automata (initialized or not) [9, 16]. The main problem discussed within the hybrid systems community has been that of proving stability and decidability of the reachability problem, and significant results have been provided for the above classes [9, 16].

It is important to bridge the gap between well assessed but independent research areas of hybrid automata and chaotic systems, and we feel that it may be worth exploring whether and how chaos can be present even in restricted classes of HA.

Let us observe that there are several ways in which chaos can originate in HA.

First of all, HA are a nondeterministic model because of these three features: the continuous evolution law $\dot{x} \in act(x)$, i.e., activity, is ruled by a differential inclusion, rather than by a differential equation (non–determinism between different time–steps); when a guard is satisfied the corresponding transition needs not be executed immediately (non–determinism between a time–step and an event–step); and finally there may be more than one transition that can occur from a given state (non–determinism between different event–steps). This non–determinism may introduce a chaotic behavior that has no counterpart in the deterministic systems considered in the literature. In this preliminary work, however, we do not explore this issue.

The goal of this paper is in showing how known *deterministic* chaotic systems can be coded in the transition structure of a HA. Thus, we will consider systems whose activities are differential equations, we will use suitable invariants to force the occurrence of a transition as soon as its guard is satisfied, and finally we will ensure that the guards of transitions are disjoint sets.

What we are left with is chaos coming out from the non–linearity of the continuous evolution and from the discontinuity (jumps) in the discrete state.

Let us first discuss the issue of non–linearity. What we are especially interested in, are special structures of the non–linearity. In particular, we consider HA whose activities within each location are linear (or affine functions) but as the the location changes these activities change as well. An example of a system that we show can be modeled in such a way is the well known Chua's circuit. This is one of the simplest systems for which the chaos has been experimentally established, numerically confirmed and mathematically proven [11]. This system is canonical in the sense that its vector field is qualitatively equivalent to a large class of 3–D vectors field; the fact that Chua's circuit family can be modeled with a very simple hybrid automaton indicates that chaos may play a major role in the hybrid systems.

A second example of an electrical system that can be modeled as an HA whose activities within each location are affine functions, is an hysteresis chaos generator [12]. The main difference with respect to the previous case, is that in this circuit chaos is generated with only two state variables. Note that this may only occur since the hysteresis jump corresponds to the third state [17].

Another feature of HA that can give rise to chaos is the discontinuity due to the jumps. We consider two examples of this kind.

The first one is a scalar function defined by Nillsen that has been shown to exhibit chaos in a weak sense [14]. The corresponding HA structure is very simple (it is a linear automaton with two continuous states, a clock and a discrete variable) but uses irrational numbers in the guard and jump relation. We also show that this system may be coded in the transition structure of a First–Order Hybrid Petri Net [3].

The second example is the logistic map. For this case, we consider two HA models. The first one, an HA whose continuous states are a clock and a discrete variable, uses the logistic map in the jump relation and thus we may say that it is explicitly chaotic. The second one implicitly encodes the logistic map in both activity, guard and jump relation.

2 Chaotic systems

As already discussed in the Introduction, there is no widely accepted definition of chaos. In the following we recall a definition of chaos firstly given by Devaney in [7]. It refers to chaos in the setting of an interval I of real numbers and a function f that maps I into I. The pair (I, f) is called a *dynamical system*, or simply a *system*. Given a system (I, f), the function $x \to f(f(x))$ also maps I into I and is denoted by $f^{[2]}$. More generally, the function $x \to f(f(\cdots f(f(x)\cdots)))$, where f appears n times, maps I into I, is denoted by $f^{[n]}$, and is called the n–th iterate of f, $f^{[1]}$ being f itself.

Thus, the set I can be seen as the set of states of a physical system and $f(x)$ may be thought of as the state of the system that results from an initial state x after one unit of time.

Whether the system (I, f) behaves chaotically is related to the behaviour of the sequence of iterates $f, f^{[2]}, f^{[3]}, \cdots, f^{[n]}, \cdots$. Roughly speaking, chaos in the system means that the iterates of f "churn up" the points of I, and this "churning up" process does not occur separately in disjoint parts of I [14].

Note that every differential equation gives rise to a map, the time one map, defined by advancing the flow one unit of time.

Now, we first provide some definition that will occur in the following, then we give the "formal" definition of chaos formulated by Devaney [7].

Definition 1 (Fixed point and periodic point). *Given a map f, if $f(x) = x$, then x is called a* fixed point *of f. If $f^{[n]} = x$ for some $n \in \mathbb{N}$, then x is called a* periodic point *of f.* ■

Definition 2 (Transitivity). *A system (I, f) is called* transitive *if, for all $\varepsilon > 0$ and $x, y \in I$, there exists $z \in I$ and $n \in \mathbb{N}$ such that $|x - z| < \varepsilon$ and $|f^{[n]}(z) - y| < \varepsilon$.* ■

Intuitively, transitivity means that we can start arbitrarily close to any state but, over time, we can attain any state to within any desired degree of approximation.

Definition 3 (Sensitivity to initial conditions). *A system (I, f) is* sensitive to initial conditions *if there is a number $\delta > 0$, such that, given any $\varepsilon > 0$ and any $x \in X$, there is $y \in X$ and a positive integer n such that $|x - y| < \varepsilon$ and $|f^{[n]}(x) - f^{[n]}(y)| > \delta$.* ■

Definition 4 ([7]). *A system (I, f) is* chaotic *if:*

(i) every point of I is the limit of a sequence whose terms are periodic points of f (in which case the periodic points of f are said to be dense *in I);*

(ii) the system is transitive;

(iii) the system is sensitive to initial conditions. ■

Note that for continuous functions the three conditions above are not independent of each other. In particular, when the function is continuous, (i) and (ii) imply (iii) [4, 8]. Moreover, if f maps I into itself, transitivity implies that there is a dense set of periodic points [5, 19].

3 Hybrid automata

A hybrid automaton consists of a classic automaton extended with a continuous state that may continuously evolve in time with arbitrary dynamics or have discontinuous jumps at the occurrence of a discrete event [13, 16]. It is a structure $H = (L, act, inv, E)$ defined as follows.

- L is a finite set of locations.

- $act : L \rightarrow Inclusions$ is a function that associates to each location $l \in L$ a differential inclusion of the form $\dot{\boldsymbol{x}} \in act_l(\boldsymbol{x}) \subseteq \mathbb{R}^n$ where the activity $act_l(\boldsymbol{x})$ is a set–valued map; if $act_l(\boldsymbol{x})$ is a singleton then it is a differential equation. A solution of a differential inclusion with initial condition $\boldsymbol{x}_0 \in \mathbb{R}^n$ is any differentiable function $\boldsymbol{\phi}(\tau)$, where $\boldsymbol{\phi} : \mathbb{R} \rightarrow \mathbb{R}^n$ such that $\boldsymbol{\phi}(0) = \boldsymbol{x}_0$ and $\dot{\boldsymbol{\phi}}(\tau) \in act_l(\boldsymbol{\phi}(\tau))$.

- $inv : L \rightarrow Invariants$ is a function that associates to each location $l \in L$ an invariant $inv_l \subset \mathbb{R}^n$. An invariant function is $\boldsymbol{x} \in inv_l$. The invariant function constrains the behaviour of the automaton state during time steps within a given subset of $\mathbb{R}^n$.

- $E \subset L \times Guards \times Jump \times L$ is the set of edges. An edge $e = (l, g, j, l') \subset E$ is an edge from location l to l' with guard g and jump relation j.

  A guard is $g \subset \mathbb{R}^n$. An edge is enabled when the state $\boldsymbol{x} \in g$.

  A jump relation is $j \subset \mathbb{R}^n \times \mathbb{R}^n$. During the jump, $\boldsymbol{x}$ is set to $\boldsymbol{x}'$ provided $(\boldsymbol{x}, \boldsymbol{x}') \in j$. When j is the identity relation, the continuous state does not change.

The state of the hybrid automaton is the pair $(l, \boldsymbol{x})$ where $l \in L$ is the discrete location, and $\boldsymbol{x} \in \mathbb{R}^n$ is the continuous state. The hybrid automaton starts from some initial state $(l_0, \boldsymbol{x}_0)$. The trajectory evolves with the location remaining constant and the continuous state $\boldsymbol{x}$ evolving within the invariant function at that location, and its first derivative remains within the differential inclusion at that location. When the continuous state satisfies the guard of an edge from location l to location l', a jump can be made to location l'. During the jump, the continuous state may get initialized to a new value $\boldsymbol{x}'$. The new state is the pair $(l', \boldsymbol{x}')$. The continuous state $\boldsymbol{x}'$ now moves within the invariant function with the new differential inclusion, followed some time later by another jump, and so on.

Several results concerning the decidability and the complexity of this model have been presented by different authors [9, 13] and many special classes of hybrid automata have been defined, namely, *Timed Automata with Skewed Clocks*, *Multi–rate Automata*, *Linear Automata* and *Rectangular Automata*. This classification originates from the requirement of determining some restricted classes for which the reachability problem can be proved to be decidable [16].

In this paper we shall deal with HA that, apart from one example, do not fit exactly into the above classes. All the same, since differences are in some cases, not too relevant, we think that it may be interesting to make them explicit. To this aim we provide these definitions.

Definition 5. *An n–dimensional* rectangle *is a set of the form $r = [l_1, u_1] \times \cdots \times [l_n, u_n] \subset \mathbb{R}^n$ with $l_i, u_i \in \mathbb{Z}_{\pm\infty}$. The i–th component of r is $r_i = [l_i, u_i]$. The set of all n–dimensional rectangles is $Rect_n$.* ■

Definition 6 ([1]). *An n–dimensional* linear automaton *is a structure $H = (L, act, inv, E)$ in which the set Inclusions contains an element $\boldsymbol{v}_l \in (\mathbb{R}^+)^n$ for each location l, i.e., at the generic location l, $\dot{x}_{i,l} = v_{i,l}$ for each i; Invariants and Guards are linear formulas[1] over the set of continuous variables, and Jump are linear terms[2] over the same set.* ■

Finally, let us introduce the following notation.

Definition 7 ([1]). *A continuous state x is called:*

- *a* clock *if $\dot{x} = 1$, $\forall\, l \in L$, and the jump relation is a reset of the form $x := \{0, 1\}$.*
- *a* discrete variable *if $\dot{x} = 0$, $\forall\, l \in L$.* ■

4 A one–to–one function with discontinuity

In this section we show how even a very simple linear automaton with a single discrete location may exhibit a chaotic behaviour. In particular we show that in this case chaos is due to the jump relations.

We consider a very simple scalar function firstly introduced by Nillsen in [14] at the aim of showing that even a function f, *one–to–one*[3] on an interval I, may lead to chaos if it has *one* point of discontinuity.

Let

$$f_\alpha(x) = \begin{cases} x + 1 - \alpha, & \text{if } 0 \le x < \alpha, \\ x - \alpha, & \text{if } \alpha \le x < 1 \end{cases}$$

[1] A *linear formula* ϕ over a set of variables V is a boolean combination of inequalities between linear terms over V.

[2] A *linear term* α over a set of variables V is a linear combination of the variables in V with rational coefficients.

[3] A function f is *one–to–one* if $f(x) = f(y)$ only if $x = y$.

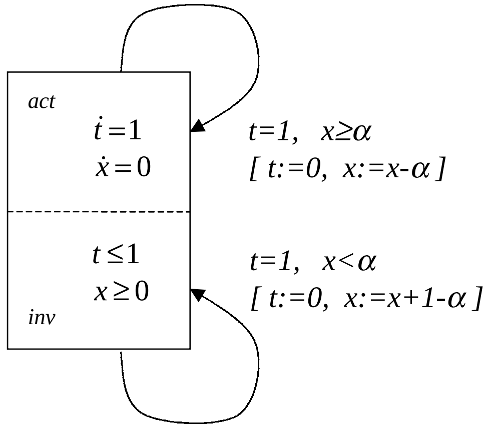


Figure 1: *The linear automaton for the dynamical system* $([0,1), f_\alpha)$.

where $\alpha \in (0,1)$.

Clearly, each function f_α is continuous and increasing on each of the intervals $[0,\alpha)$ and $[\alpha,1)$, but is discontinuous at α.

In [14] the author showed that whenever α is an irrational number, then the dynamical system $([0,1), f_\alpha)$ has no periodic points, but it is transitive and sensitive to initial conditions, thus exhibiting a form of chaotic behaviour, although in a slightly weaker sense than that of Devaney [7]. More precisely, even if f_α has no dense set of periodic points, it *does* produce a dense set of "approximately periodic" points [14].

Now, let us show how the dynamical system $([0,1), f_\alpha)$ can be described by the hybrid automaton $H = (L, act, inv, E)$ in figure 1.

It is a very simple linear automaton with a single location, but uses irrational numbers in the guard and jump relation. Its continuous states are a clock (t) and a discrete variable (x), being the activity a single point with constant rates equal to 1 and 0, respectively, and the jump relation of t a reset to zero. More precisely,

- $L = \{l\}$;
- $act_l = [1,\ 0]^T$;
- $inv_l = \{t,\ x \mid t \leq 1,\ x \geq 0\}$;
- $E = \{(l, \{t,\ x \mid t = 1, x \geq \alpha\}, [t := 0,\ x := x - \alpha], l),\ (l, \{t,\ x \mid t = 1, x < \alpha\}, [t := 0,\ x := x + 1 - \alpha], l)\}$.

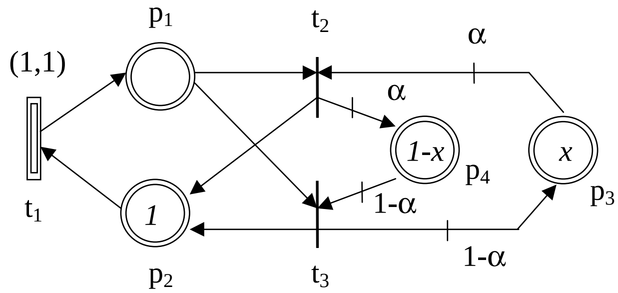


Figure 2: *Hybrid Petri net for the logistic map.*

Note that x is the state variable that presents a chaotic behaviour. It assumes a sequence of numerical values that is the same as that of the dynamical system $([0,1), f_\alpha)$. The state variable t has been introduced so as to impose the updating of x as soon as t becomes unitary.

Thus, the chaotic behavior of the above hybrid automaton only originates from the jump relations.

Now, let us briefly discuss how the same chaotic behaviour can also be exhibited by an hybrid Petri net. We refer to the untimed version of the First–Order Hybrid Petri net model presented by Balduzzi *et al.* in [3].

The hybrid Petri net system is shown in figure 2 where the marking is representative of the initial condition. The firing of the continuous transition t_1 represents the time evolving. During the time interval Δt, $\Delta m(p_1) = -\Delta m(p_2) = v \cdot dt$ where $v \in (V_{min}, V_{max}) = (1,1)$, i.e., being $V_{min} = V_{max} = 1$, transition t_1 should always be enabled and may only fire at a unitary firing speed. Thus, initially t_1 is the only enabled transition and its firing produces an increasing in the marking of p_1 and a decreasing in the marking of p_2, both with unitary constant slope. As soon as p_2 is empty, t_1 cannot fire, thus either t_2 or t_3 should fire, thus resetting p_1 and p_2 to 0 and 1, respectively.

In particular, if $x < \alpha$, then transition t_3 fires, thus producing and increasing of $1 - \alpha$ in the marking of p_3. On the contrary, if $x \geq \alpha$, the transition t_2 fires, thus producing a decreasing of α in the marking of p_3.

It is evident that the chaotic behaviour is exhibited by the marking of place p_3. The continuous place p_4 is only a co–buffer, i.e., during all the net evolution it holds that $m(p_3) + m(p_4) = 1$.

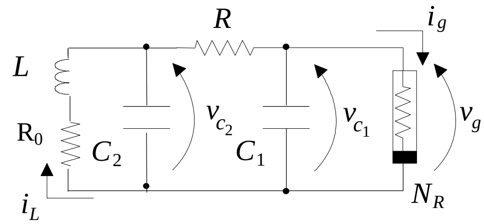


Figure 3: *Chua's oscillator.*

5 Chua's circuit and topologically conjugate systems.

In this section we consider Chua's circuit and the family of topologically conjugate systems. We show how their non–linear dynamics can be modeled by an HA where the activities within each location are linear or affine functions.

Chua's circuit consists of a linear inductor L, two linear capacitors C_1 and C_2, a linear resistor R and a voltage–controlled PWL resistor N_R. By adding a linear resistor in series with the inductor, we obtain the Chua's oscillator [10] shown in figure 3. The oscillator is completely described by a system of three ordinary differential equations. By a change of variables, the dimensionless state equations of Chua's oscillator becomes

$$\begin{cases} dx_1/dt = k\alpha(x_2 - x_1 - h(x_1)) \\ dx_2/dt = k(x_1 - x_2 + x_3) \\ dx_3/dt = k(-\beta x_2 - \gamma x_3) \\ h(x_1) = bx_1 + (a-b)\,[\,|x_1+1| - |x_1-1|\,]\,/2 \end{cases} \tag{1}$$

where

$$\begin{array}{lll} x_1 \equiv v_{c_1}/E, & x_2 \equiv v_{c_2}/E, & x_3 \equiv i_L R/E, \\ \alpha \equiv C_2/C_1, & \beta \equiv R^2 C_2/L, & \gamma \equiv RR_0C_2/L \\ a \equiv RG_a, & b \equiv RG_b, & t \equiv \tau/|RC_2|, \\ \text{and} & & \\ k = 1 & \text{if} & RC_2 > 0, \\ k = -1 & \text{if} & RC_2 < 0. \end{array} \tag{2}$$

Chua's oscillator is *canonical* in the sense that its vector field is *topologically conjugate* (i.e. qualitatively equivalent) to a large class $\mathcal{C}$ of 3–D vector fields [6].

Definition 8. *A dynamical system defined by a state equation*

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^3 \tag{3}$$

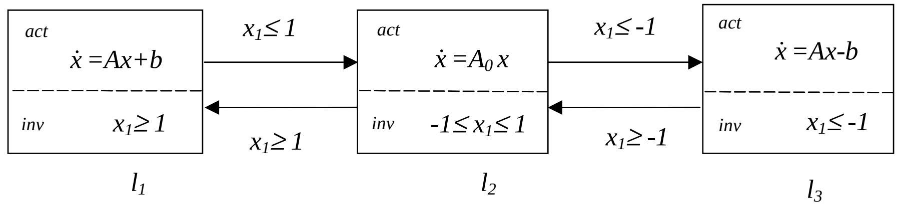


Figure 4: *The HA for Chua's oscillator family.*

is said to belong to class $\mathcal{C}$ iff

1. $\boldsymbol{f}(\cdot)$ *is continuous,*
2. $\boldsymbol{f}(\cdot)$ *is odd–symmetric, i.e.,* $\boldsymbol{f}(\boldsymbol{x}) = -\boldsymbol{f}(-\boldsymbol{x})$*,*
3. $\mathbb{R}^3$ *is partitioned by* 2 *parallel boundary planes* U_1 *and* U_{-1} *into an inner region* D_0 *containing the origin, and two outer regions* D_1 *and* D_{-1}*, and* $\boldsymbol{f}(\cdot)$ *is affine in each region*[4]*.* ■

Without loss of generality, the boundary planes U_1 and U_{-1} can be chosen to be

$$\begin{aligned} U_1 &: x_1 = 1, \\ U_{-1} &: x_1 = -1. \end{aligned} \tag{4}$$

Then any vector field in the family $\mathcal{C}$ can be represented as

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b} & x_1 &\geq 1 \\ \dot{\boldsymbol{x}} &= \boldsymbol{A}_0\boldsymbol{x} & -1 \leq x_1 &\leq 1 \\ \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} & x_1 &\leq 1 \end{aligned} \tag{5}$$

where $\boldsymbol{A}, \boldsymbol{A}_0 \in \mathbb{R}^{3\times 3}$, $\boldsymbol{b} \in \mathbb{R}^{3\times 1}$ and because of continuity

$$\boldsymbol{A}_0 = \boldsymbol{A} + \begin{bmatrix} \boldsymbol{b} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}.$$

Now, we show how the Chua's oscillator, as well as all dynamical systems in $\mathcal{C}$ (see definition 8), can be described by the hybrid automaton $H = (L, act, inv, E)$ reported in figure 4. It can be immediately seen that the activities are single points, (i.e., linear differential equations rather than inclusions) and guards and invariants are half planes. More precisely,

[4]By condition 2, the vector field D_0 must be linear, i.e., it is zero at the origin.

- $L = \{l_1, l_2, l_3\}$;
- $act_{l_1} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$, $act_{l_2} = \boldsymbol{A}_0\boldsymbol{x}$, $act_{l_3} = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}$;
- $inv_{l_1} = \{\boldsymbol{x} \mid x_1 \geq 1\}$, $inv_{l_2} = \{\boldsymbol{x} \mid -1 \leq x_1 \leq 1\}$, $inv_{l_3} = \{\boldsymbol{x} \mid x_1 \leq -1\}$;
- $E = \{(l_1, \{\boldsymbol{x} \mid x_1 \leq 1\}, id, l_2),\ (l_2, \{\boldsymbol{x} \mid x_1 \geq 1\}, id, l_1),\ (l_2, \{\boldsymbol{x} \mid x_1 \leq -1\}, id, l_3),\ (l_3, \{\boldsymbol{x} \mid x_1 \geq -1\}, id, l_2)\}$.

6 An hysteresis chaos generator

In this section we consider an extremely simple example of hysteresis chaos generator [12] whose chaos generation is guaranteed theoretically [18]. The circuit is reported in figure 5 and is designed by using only six elements: two capacitors, two resistors, one linear voltage controlled current source (VCCS) and one hysteresis VCCS.

The circuit dynamics is defined by a system of two ordinary differential equations [12], that can also be written in a normalized form, by introducing appropriate dimensionless variables and parameters:

$$\begin{cases} dx_1/dt = -(x_1 + x_2) \\ dx_2/dt = -(1 - RG)x_1 - (2 - RG)x_2 - p\,h(x_1) \end{cases}$$

where $t = \tau/RC$ is a normalized measure of time, $x_1 = v_{c_1}/V$, $x_2 = v_{c_2}/V$, $p = rI_0/V$ and $h(x_1) \equiv -I_h(Vx_1)/I_0$ is a normalized hysteresis as shown in figure 6: $h(x_1)$ is switched from 1 to -1 if x_1 hits -1 and vice versa.

This system can be modeled by the hybrid automaton $H = (L, act, inv, E)$ in figure 7 where $x \in \mathbb{R}^2$, $A \in \mathbb{R}^{2\times 2}$, $b \in \mathbb{R}^2$.

It is a very simple HA whose structure is quite similar to that relative to the Chua's oscillator family:

- $L = \{l_1, l_2\}$;
- $act_{l_1} = Ax - b$, $act_{l_2} = Ax + b$;
- $inv_{l_1} = \{x_1 \geq -1\}$, $inv_{l_2} = \{x_1 \leq 1\}$;

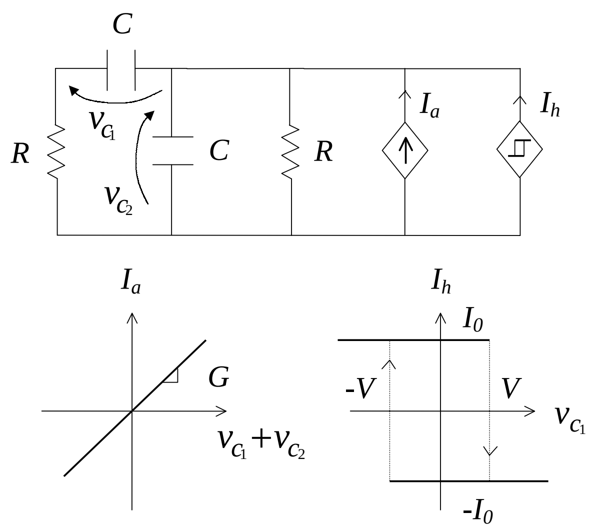


Figure 5: *Hysteresis chaos generator.*

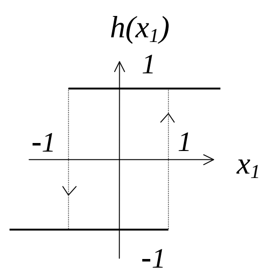


Figure 6: *Normalized hysteresis.*

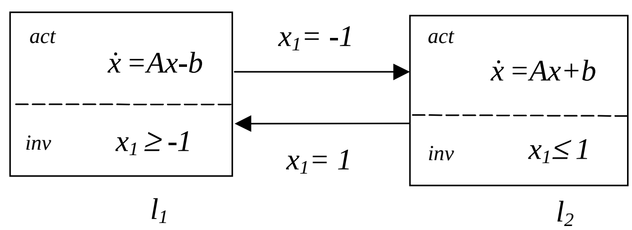


Figure 7: *The HA for the hysteresis chaos generator.*

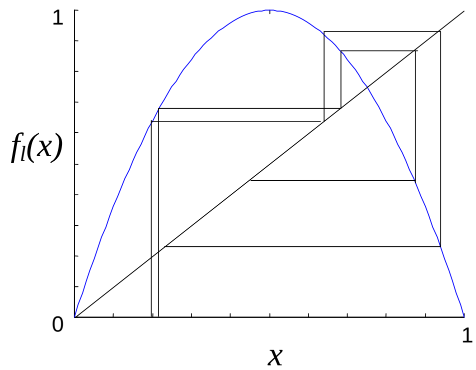


Figure 8: *Logistic map — divergence of nearby orbits.*

- $E = \{(l_1, \{x \mid x_1 = -1\}, id, l_2), (l_2, \{x \mid x_1 = 1\}, id, l_1)\}$.

Location l_1 corresponds to $h(x_1) = 1$, i.e., $x_1 \geq -1$, while location l_2 corresponds to $h(x_1) = -1$, i.e., $x_1 \leq 1$. Thus, as $h(x_1)$ is switched from 1 to -1 if x_1 hits -1 and vice versa, analogously the activity hits from $act_{l_1} = Ax - b$ to $act_{l_2} = Ax + b$ if x_1 hits -1 and vice versa.

The main difference with respect to the HA in figure 4 is that in this case we have only two discrete locations and above all, two continuous states.

As it is well known, in autonomous systems at least three states are required to generate chaos. However, in this circuit, besides two states of capacitor voltages, the hysteresis jump corresponds to the third state [17].

7 The logistic map

One of the simplest maps that exhibits chaotic behaviour is the logistic map:

$$f_l(x) = 4x(1-x), \quad 0 < x < 1.$$

The logistic map is one of the most famous examples of exponential divergence of nearby trajectories as shown in figure 8, where the trajectories from two nearby initial conditions are seen to be diverging already after three iterations.

The dynamical system $((0,1), f_l)$ can be described by two different hybrid automata.

The first one, sketched in figure 9.a, is a HA with a clock (t) and a discrete variable (x), where the chaotic behaviour is due to the jump relation that happens as soon as t becomes unitary. In this case the structure of the hybrid automaton is analogous to that already presented in section 4, apart from the single edge $e \in E$ and the non–linear jump relation of x. Thus, it is not discussed further.

On the contrary, the second one, reported in figure 9.b, is an hybrid automaton that belongs to none of the above mentioned classes. More precisely,

- $L = \{l_1, l_2\}$;
- $act_{l_1} = [1,\ 0,\ 4(1-x)]^T$ and $act_{l_2} = [1,\ 0,\ 0]^T$;
- $inv_{l_1} = \{t,\ x,\ y \mid t \le x,\ x, y \ge 0\}$ and $inv_{l_2} = \{t,\ x,\ y \mid t \le 1,\ x, y \ge 0\}$;
- $E = \{(l_1, \{t,\ x,\ y \mid t = x\}, id, l_2), (l_2, \{t,\ x,\ y \mid t = 1\}, [t := 0,\ x := y,\ y = 0], l_1)\}$.

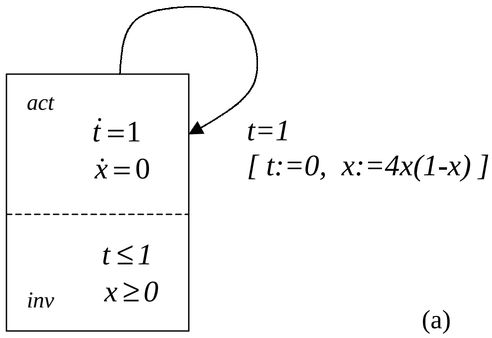
The state variable that presents a chaotic behaviour is the discrete variable x, whose sequence of numerical values is the same as that of the dynamical system $((0,1), f_l)$. In fact, it keeps constant when the system evolves within both the discrete locations l_1 and l_2. Within the discrete location l_1 the continuous variable y evolves as $y(t) = 4(1-x)t$, while it keeps constantly equal to $4x(1-x)$ within the discrete location l_2. Finally, when $t = 1$, x is updated to y, while both t and y are reset to 0.

Note that the same chaotic sequence of x numerical values could also have been obtained by removing the discrete location l_2 from the previous automaton and assuming a single edge from l_1 to l_1 itself. In such a case the guard would have been $t = x$ and the jump relation $t, y := 0,\ x := y$. However, in such a case the time interval between two consecutive jumps would have been non–constant, but dependent on the current value of x.

Let us finally observe that the main interest in this hybrid automaton lies in the fact that, while in the previous examples, chaos is only related to either the jump relations or to the activity sets, in this case chaos originates from the combination of both these two elements.

8 Conclusions

In this paper we have shown, via some significant examples, how deterministic chaotic systems can be coded in the transition structure of an hybrid automaton.



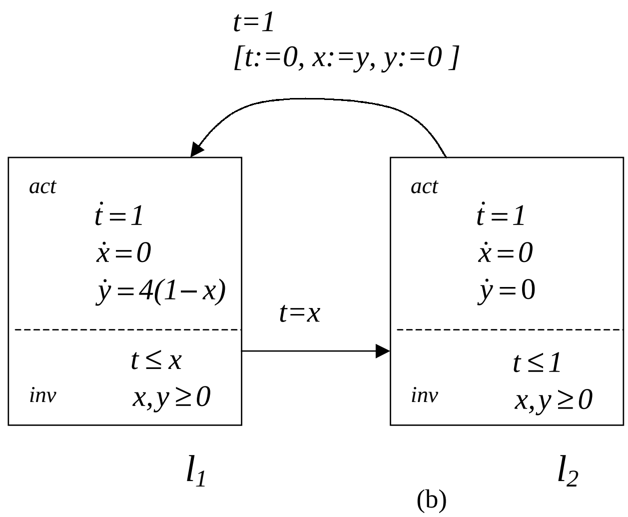


Figure 9: *Hybrid automata for the logistic map.*

We have first considered a scalar function that exhibits chaotic behaviour in a weak sense. We have shown that the same behaviour may also be exhibited by the discrete variable of a linear automaton, provided an irrational number is introduced in the guard and jump relation.

Also Chua's circuit family and a normalized hysteresis chaos generator have been modeled by HA, but in both these cases chaos comes out from the different activities (linear and affine functions) within the discrete locations.

Finally, we have presented an example (the logistic map) where chaos originates from the combination of both the above elements, namely, jump relations and activity sets.

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